S176. Let ABC be a triangle and let AA_1, BB_1, CC_1 be cevians intersecting at P. Denote by $K_a = K_{AB_1C_1}$, $K_b = K_{BC_1A_1}$, $K_c = K_{CA_1B_1}$. Prove that $K_{A_1B_1C_1}$ is a root of the equation

$$x^3 + (K_a + K_b + K_c)x^2 - 4K_aK_bK_c = 0.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

First solution by Arkady Alt, San Jose, California, USA

Without loss of generality assume that area of triangle ABC is 1. Let p_a, p_b, p_b be the baricentric coordinates of P, that is $p_a, p_b, p_c > 0, p_a + p_b + p_c = 1$ and

$$p_a\overrightarrow{PA} + p_b\overrightarrow{PB} + p_c\overrightarrow{PC} = 0.$$

Since $\dfrac{AC_1}{BC_1} = \dfrac{p_b}{p_a}$ and $\dfrac{[ACC_1]}{[BCC_1]} = \dfrac{AC_1}{BC_1}$ then

$$[ACC_1] = \frac{p_b}{p_a + p_b}[ABC] = \frac{p_b}{p_a + p_b}.$$

Also, $\dfrac{[AB_1C_1]}{[CB_1C_1]} = \dfrac{AB_1}{CB_1} = \dfrac{p_c}{p_a}$ yields

$$\frac{[AB_1C_1]}{[ACC_1]} = \frac{p_c}{p_c + p_a}.$$

Hence, $K_a = [AB_1C_1] = \dfrac{p_bp_c}{(p_a + p_b)(p_c + p_a)}$. Similarly,

$$K_b = \frac{p_cp_a}{(p_b + p_c)(p_a + p_b)},\ K_c = \frac{p_ap_b}{(p_c + p_a)(p_b + p_c)}.$$

Let $K = [A_1B_1C_1]$, then

$$\begin{aligned} K &= [ABC] - ([AB_1C_1] + [BC_1A_1] + [CA_1B_1]) \\ &= 1 - (K_a + K_b + K_c) = 1 - \sum_{cyc}\frac{p_bp_c}{(p_a + p_b)(p_c + p_a)} \\ &= \frac{2p_ap_bp_c}{(p_a + p_b)(p_b + p_c)(p_c + p_a)}. \end{aligned}$$

On the other hand, since

$$\frac{K_a}{K} = \frac{p_b + p_c}{2p_a},\ \frac{K_b}{K} = \frac{p_c + p_a}{2p_b},\ \frac{K_c}{K} = \frac{p_a + p_b}{2p_c}$$

then

$$\frac{K_aK_bK_c}{K^3} = \frac{(p_a + p_b)(p_b + p_c)(p_c + p_a)}{8p_ap_bp_c} = \frac{1}{4K}.$$

Thus, $K^2 = 4K_aK_bK_c$ and, therefore,

$$\begin{aligned} K = 1 - (K_a + K_b + K_c) &\iff K^3 = K^2 - (K_a + K_b + K_c)\, K^2 \\ &\iff K^3 + (K_a + K_b + K_c)\, K^2 - 4K_aK_bK_c = 0. \end{aligned}$$

Second solution by Daniel Campos Salas, Costa Rica

Let x, y, z be the areas of the triangles BPC, CPA, APB, respectively. It's easy to prove that $\frac{AB_1}{B_1C} = \frac{z}{x}$ and $\frac{AC_1}{C_1B} = \frac{y}{x}$. Therefore we have that

$$K_a = \frac{1}{2}AB_1 \cdot AC_1 \sin A = \frac{1}{2} \cdot \frac{zAC}{x+z} \cdot \frac{yAB}{x+y} \sin A = \frac{yz(x+y+z)}{(x+y)(x+z)}.$$

The expressions for K_b and K_c are obtained analogously. It follows that $K_{A_1B_1C_1}$ equals

$$\begin{aligned} & (x+y+z)\left(1 - \frac{yz}{(x+y)(x+z)} - \frac{zx}{(y+z)(y+x)} - \frac{xy}{(z+x)(z+y)}\right) \\ = & \frac{2xyz(x+y+z)}{(x+y)(y+z)(z+x)}. \end{aligned}$$

Finally, note that

$$\begin{aligned} & K^3_{A_1B_1C_1} + K_{A_1B_1C_1}(K_a + K_b + K_c) \\ = & K^2_{A_1B_1C_1}(K_{A_1B_1C_1} + K_a + K_b + K_c) \\ = & \left(\frac{2xyz(x+y+z)}{(x+y)(y+z)(z+x)})\right)^2 (x+y+z) \\ = & \frac{4(xyz)^2(x+y+z)^3}{((x+y)(y+z)(z+x))^2} \\ = & 4K_aK_bK_c, \end{aligned}$$

which proves that $K_{A_1B_1C_1}$ is a root of the given polynomial.

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.